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Spatial Analyticity of the Solutions to the Subcritical Dissipative Quasi-geostrophic Equations

HONGJIE DONG & DONG LI

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Abstract

We employ a new bilinear estimate to show that solutions to the subcritical dissipative quasi-geostrophic equations with initial data in the scaling-invariant Lebesgue space are analytic in space variables. Some decay in time estimates for space–time derivatives are also obtained.

1. Introduction and the main theorem

We are interested in the initial value problem of the two-dimensional dissipative quasi-geostrophic equations

$$\begin{cases} \theta_t + u\nabla\theta + (-\Delta)^{\gamma/2}\theta = 0, & \text{on } \mathbb{R}^2 \times (0,\infty), \\ \theta(0,x) = \theta_0(x), & x \in \mathbb{R}^2, \end{cases} \tag{1}$$

where $\gamma \in (0, 2]$ is a fixed parameter and the velocity $u = (u_1, u_2)$ is divergence free and determined by the Riesz transforms of the potential temperature θ:

$$u = (-\mathcal{R}_2\theta, \mathcal{R}_1\theta) = (-\partial_{x_2}(-\Delta)^{-1/2}\theta, \partial_{x_1}(-\Delta)^{-1/2}\theta).$$

This equation is an important model in geophysical fluid dynamics. It is derived from the general quasi-geostrophic equations in the special case of constant potential vorticity and buoyancy frequency. When $\gamma = 1$, it is the dimensionally correct analogue of the three-dimensional Navier–Stokes equations. Recently, this equation has been studied intensively; see [5–9, 19], and references therein.

The global existence of a weak solution to (1) follows from RESNICK [16]. The cases $\gamma > 1$, $\gamma = 1$, and $\gamma < 1$ are called subcritical, critical, and supercritical respectively. The subcritical case is the best understood. WU [19] established the global existence of a unique regular solution to (1) with initial data θ_0 in L^p for $p > 2/(\gamma - 1)$. With initial data in the scaling-invariant space $L^{2/(\gamma-1)}$, the proof of global well-posedness can be found, for example, in the recent work of CARRILLO and FERREIRA [1], where the asymptotic behavior of the solutions is also studied. By using the Fourier splitting method, CONSTANTIN and WU [5] showed the global existence of a regular solution on the torus with periodic boundary conditions and also a L^2 decay estimate for weak solutions with data in $L^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$. JU [8,9] shows local well-posedness for all cases with initial data in the Sobolev space H^s for $s > 2 - \gamma$, and global in time well-posedness for the subcritical case. These results were later improved upon and extended to all three cases with initial data in $H^{2-\gamma}$ by JU [10], and independently by MIURA [13] who uses a different approach. We also note that an important maximum principle of solutions to (1) was established by CÓRDOBA and CÓRDOBA [7]. It is already well known that in the critical case the solutions are smooth. However, it seems to us that higher regularity and analyticity properties of these solutions have not been studied in detail.

In this paper, we study the subcritical quasi-geostrophic (1) with initial data in $L^{2/(\gamma-1)}(\mathbb{R}^2)$. Clearly, if $\theta(t,x)$ is a solution, then

$$\theta_\lambda(t,x) = \lambda^{\gamma-1}\theta(\lambda^\gamma t, \lambda x)$$

is also a solution of (1) with initial data $\theta_\lambda(0,x) = \lambda^{\gamma-1}\theta_0(\lambda x)$. The Lebesgue space $L^{2/(\gamma-1)}(\mathbb{R}^2)$ is called scaling invariant because

$$\|\theta_\lambda(0,\cdot)\|_{L^{2/(\gamma-1)}} = \|\theta_0(\cdot)\|_{L^{2/(\gamma-1)}}$$

for any $\lambda > 0$. As we mentioned before, local well-posedness in $L^{2/(\gamma-1)}(\mathbb{R}^2)$ is well known and can be proved by Kato's method (see, for example, the Appendix of [1]). The global in time well-posedness then follows from the local well-posedness and the L_p maximum principle [7]. We are interested in the higher regularity of these solutions. We establish an optimal estimate of the solutions in higher-order Sobolev norms (Theorems 1, 2). This estimate then implies that the solutions are analytic in space variables (see Remark 2). A decay in time estimate for any space derivatives is also obtained (Theorem 2). Note that similar analyticity rate estimates were established for the Navier–Stokes equations in [14,17]. However, the existing methods do not give analyticity of the solutions for $\gamma < 2$ (see Remark 7).

Before we state our results precisely, we shall explain some notations and settings. Define the kernel $G(t,x) = G_\gamma(t,x)$ by its Fourier transform

$$\widehat{G_\gamma}(t,\xi) = \mathrm{e}^{-t|\xi|^\gamma} \quad \text{for } t > 0.$$

Then $G(t,x)$ is the fundamental solution of the linear operator $\partial_t + (-\Delta)^{\gamma/2}$. Clearly, $G_2(t,x)$ is the heat kernel and $G_1(t,x)$ is the Poisson kernel. Moreover, $G_\gamma(t,x)$ has the scaling property

$$G_\gamma(t,x) = t^{-\frac{2}{\gamma}} G_\gamma\left(1, t^{-\frac{1}{\gamma}} x\right). \tag{2}$$

It is well known that (1) can be rewritten as an integral equation

$$\theta(t) = G(t) * \theta_0 - \int_0^t G(t-s,\cdot) * (u\nabla\theta)(s,\cdot)\,\mathrm{d}s. \tag{3}$$

For $q \in (\frac{2}{\gamma-1}, \infty]$, $T \in (0, \infty]$, let $\alpha = 1 - \frac{1}{\gamma} - \frac{2}{q\gamma}$. We define the auxiliary space (first introduced by WEISSLER [18] in the study of mild solutions of the Navier–Stokes equations),

$$K_q = \left\{ u \mid t^\alpha u \in BC((0, T), L^q_x), \quad \lim_{t \to 0} t^\alpha \|u\|_{L^q_x} = 0 \right\}.$$

Introduce the Banach spaces,

$$X^0_{q,T} = BC\left([0, T), L_x^{\frac{2}{\gamma-1}}\right) \cap K_q$$

with norm

$$\|u\|_{X^0_{q,T}} = \max\left\{ \|u\|_{L^\infty_t([0,T))L_x^{\frac{2}{\gamma-1}}}, \|t^\alpha u\|_{L^\infty_t((0,T))L^q_x} \right\},$$

and

$$X^k_{q,T} = X^{k-1}_{q,T} \cap \left\{ u \mid t^{\frac{k}{\gamma}} D^k u \in BC\left((0, T), L_x^{\frac{2}{\gamma-1}}\right) \right\}$$
$$\cap \left\{ u \mid t^{\frac{k}{\gamma}+\alpha} D^k u \in BC((0, T), L^q_x) \right\}$$

with norm

$$\|u\|_{X^k_{q,T}} = \max\left\{ \|u\|_{X^{k-1}_{q,T}}, \|u\|_{Y^k_{q,T}} \right\},$$

where $\| \cdot \|_{Y^k_{q,T}}$ is a semi-norm defined by

$$\|u\|_{Y^k_{q,T}} = \max\left\{ \|t^{\frac{k}{\gamma}} D^k u\|_{L^\infty_t((0,T))L_x^{\frac{2}{\gamma-1}}}, \|t^{\frac{k}{\gamma}+\alpha} D^k u\|_{L^\infty_t((0,T))L^q_x} \right\}.$$

The main motivation of the article is as follows. Since the solutions to the corresponding linear equations are spatial analytic if $\gamma \geqq 1$, it is natural to ask whether this is still true for the solutions to (1). Here we give an affirmative answer to this question for $\gamma \in (1, 2]$. Next we state our main results.

Theorem 1 (Short-time spatial analyticity). *Let $\gamma \in (1, 2]$, $\theta_0 \in L^{\frac{2}{\gamma-1}}$ and $q \in (\frac{2}{\gamma-1}, \infty]$. Then there exists $T > 0$ such that the initial value problem for* (1) *has a unique mild solution $\theta(t, x)$ in $X^k_{q,T}$ for any integer $k \geqq 0$, and we have the estimate*

$$\|\theta\|_{X^k_{q,T}} \leqq C^{k+1} k^k \tag{4}$$

with a constant C independent of k and T.

Remark 1. The reason we shall use the so-called two-norm approach is because the bilinear operator is not continuous from $C([0, T), L_x^{\frac{2}{\gamma-1}}) \times C([0, T), L_x^{\frac{2}{\gamma-1}})$ to $C([0, T), L_x^{\frac{2}{\gamma-1}})$. However, the mild solution is unique in the space $C([0, T), L_x^{\frac{2}{\gamma-1}})$ (see [4]).

The most important part in the proof of Theorem 1 is Lemma 3 (or Lemma 6 in the second proof). Our treatment of the bilinear term is novel and makes better use of the smoothing effect of the Stokes kernel. The idea is that the smoothing effect takes place gradually in time (see Section 3).

From Theorem 1 and the L^p maximum principle for the quasi-geostrophic equations, it is not difficult to prove the long-time spatial analyticity of $\theta(t, x)$.

Corollary 1. *Under the assumptions of Theorem 1, the initial value problem for (1) has a unique mild solution $\theta(t, x)$ in $X^k_{q,\infty}$ for any integer $k \geqq 0$. Moreover, for any $T > 0$ we have the estimate*

$$\|\theta\|_{X^k_{q,T}} \leqq C_T C^k k^k, \tag{5}$$

with constants C_T, C independent of k.

The next theorem improves the previous corollary by removing the time-dependent constant C_T.

Theorem 2 (Long-time spatial analyticity). *Under the assumptions of Theorem 1, the initial value problem for (1) has a unique mild solution $\theta(t, x)$ in $X^k_{q,\infty}$ for any integer $k \geqq 0$, and we have the estimate*

$$\|\theta\|_{X^k_{q,\infty}} \leqq C^{k+1} k^k \tag{6}$$

with a constant C independent of k. In particular, the following decay in time estimate holds

$$\|D^k \theta(t, \cdot)\|_{L^q_x} \leqq C^k t^{-\frac{k}{\gamma} - \alpha} k^k.$$

Remark 2. Taking $q = \infty$ in Theorem 2 (or Corollary 1) implies that $\theta(t, x)$ is spatial analytic for any $t \in (0, \infty)$. Moreover, the radius of (spatial) analyticity of the solution $\theta(t, x)$ increases with time at a rate proportional to $t^{1/\gamma}$.

Remark 3. Note that in Theorem 2 we do not impose any smallness assumption on the initial data.

Remark 4. In fact, our proof gives a stronger result (see Section 7): for any integer $k \geqq 0$ and any real number $\beta \geqq 0$, $q \in (\frac{2}{\gamma - 1}, \infty]$, we have

$$\|D^k (-\Delta)^{\beta/2} \theta(t, \cdot)\|_{L^q_x} \leqq C^{k+\beta} t^{-\frac{k+\beta}{\gamma} - \alpha} \Gamma(k + \beta + 1).$$

Moreover, the same estimate is true with u in place of θ. Consequently, u is also spatial analytic.

For the mixed space–time derivatives, we have the following estimate.

Theorem 3 (Bound on mixed space–time derivatives). *Let $\gamma \in (1, 2]$, $\theta_0 \in L^{\frac{2}{\gamma-1}}$. Let $\theta(t, x)$ be the unique global solution in Theorem 2. Then there exists a constant $C = C(\theta_0, \gamma)$ such that*

(i) *for any* $0 < t \leq 1$*, we have*

$$\left\| \partial_t^m \partial_x^k \theta \right\|_{L_x^\infty} \leq C^{2m+k+1} \cdot (2m+k)^{2m+k} \cdot t^{-\frac{2m+k+1}{\gamma}}, \tag{7}$$

(ii) *for any* $t \geq 1$*, the following decay in time estimate holds*

$$\left\| \partial_t^m \partial_x^k \theta \right\|_{L_x^p} \leq C^{m\gamma+k+1} \cdot (2m+k)^{2m+k} \cdot t^{-\frac{k}{\gamma}-m-\alpha}, \tag{8}$$

where $\alpha = 1 - \frac{1}{\gamma} - \frac{2}{p\gamma}$*. Moreover, for any* $\varepsilon > 0$*, there exists a constant* $C_\varepsilon = C_\varepsilon(\theta_0, \gamma)$*, such that*

$$\left\| \partial_t^m \partial_x^k \theta \right\|_{L_x^\infty} \leq C_\varepsilon^{m\gamma+k+1} \cdot (2m+k)^{2m+k} \cdot t^{-\left(\frac{k-1}{\gamma}+m+1-\varepsilon\right)}. \tag{9}$$

Remark 5. Without much more work, we can modify our proof to show the following limiting behavior of θ as $t \to 0$:

$$\lim_{t \to 0} t^{\frac{k}{\gamma}+\alpha} \| D^k \theta(t, \cdot) \|_{L_x^q} = 0, \tag{10}$$

$$\lim_{t \to 0} t^{\frac{2m+k+1}{\gamma}} \left\| \partial_t^m \partial_x^k \theta(t, \cdot) \right\|_{L_x^\infty} = 0, \tag{11}$$

for any $k \geqq 0$, $m \geqq 0$, $q \in [\frac{2}{\gamma-1}, \infty]$, and $(k, q) \neq (0, \frac{2}{\gamma-1})$.

Remark 6. These results can be easily generalized to higher dimensional quasi-geostrophic equations with obvious modifications.

The remaining part of the article is organized as follows. In the next section, we recall some estimates of the kernel. The bilinear term is estimated in Section 3, where we make a novel use of the smoothing effect of the Stokes kernel. Sections 4 and 5 are devoted to the existence of solutions in $X_{q,T}^k$, $k = 0, 1, 2, \ldots$ and an analyticity rate estimate. We then give the proof of Theorems 1 and 2 in Section 6. The proof of Theorem 2 uses a trick of translation of the time. A second proof of it based on fractional bootstraping is provided in Section 7. Theorem 3 will be proved in Section 8, and in the last section we provide a remark on the case with periodic boundary conditions.

2. Estimates of the linear term

We need the following refined estimates of the linear term.

Lemma 1. *Let* $\gamma \in (0, \infty)$*,* $1 \leqq r \leqq p \leqq \infty$*. For any integer* $k \geqq 0$ *and* $\theta_0 \in L^r(\mathbb{R}^2)$*,*

$$\| D_x^k G(t, \cdot) \|_{L_x^p} \leqq C_\gamma^{k+1} k^{\frac{k}{\gamma}} t^{-\frac{k}{\gamma}-\frac{2}{\gamma}\left(1-\frac{1}{p}\right)}, \tag{12}$$

$$\| D_x^k G(t, \cdot) * \theta_0(\cdot) \|_{L_x^p} \leqq C_\gamma^{k+1} k^{\frac{k}{\gamma}} t^{-\frac{k}{\gamma}-\frac{2}{\gamma}\left(\frac{1}{r}-\frac{1}{p}\right)} \| \theta_0 \|_{L^r}, \tag{13}$$

where $C_\gamma \geqq 1$ *is a constant depending only on* γ. *Moreover, for any* $\theta_0 \in L^{\frac{2}{\gamma-1}}$, $\gamma \in (1, 3]$, *and* $q \in (\frac{2}{\gamma-1}, \infty]$, *we have*

$$\sup_{t\in(0,\infty)} t^\alpha \|G(t,\cdot) * \theta_0(\cdot)\|_{L^q_x} \leqq C_\gamma \|\theta_0\|_{L^{\frac{2}{\gamma-1}}}, \tag{14}$$

$$\lim_{t\to 0^+} t^\alpha \|G(t,\cdot) * \theta_0(\cdot)\|_{L^q_x} = 0, \tag{15}$$

where $\alpha = 1 - \frac{1}{\gamma} - \frac{2}{\gamma q}$.

Proof. Notice that (13) follows from (12) and Hölder's inequality, (14) is a special case of (13), and the proof (15) of can be found, for example, in Lemma 2.1 of [1]. It suffices to prove the estimate (12). However, this is an easy consequence of the scaling property (2) if $k = 0, 1$. For $k \geqq 2$, (12) follows from the identity

$$D_x^k G(t,\cdot) = D_x G\left(\frac{t}{2k},\cdot\right) * \cdots * D_x G\left(\frac{t}{2k},\cdot\right) * G\left(\frac{t}{2},\cdot\right),$$

and the Young's inequality. □

Similarly we can prove the following fractional linear estimate.

Lemma 2. *Let* $\gamma \in (0, \infty)$, $1 \leqq p \leqq \infty$. *Let* $k \geq 0$ *be an integer and* $\varepsilon \in [0, 1]$. *Then for some constant* $C_\gamma > 0$, *we have*

$$\left\|\Lambda^{k+\varepsilon} G(t,\cdot)\right\|_{L^p_x} \leq C_\gamma^{k+1} k^{\frac{k}{\gamma}} t^{-\frac{k+\varepsilon}{\gamma} - \frac{2}{\gamma}\left(1-\frac{1}{p}\right)}. \tag{16}$$

Proof. This is the same as in the proof of Lemma 1. One fact used is that $\Lambda^\varepsilon G(t,\cdot) \in L^p_x$ for any $1 \leq p \leq \infty$. For $\varepsilon \in [0, 1]$, this easily follows from the following pointwise estimate (see Lemma 2.2 of [12]),

$$\left|\Lambda^\varepsilon G(1,x)\right| \leq C \cdot (1 + |x|)^{-2-\varepsilon}.$$

The lemma is proved. □

3. Estimates of the bilinear term

Since u is divergence free, we have $u\nabla\theta = \nabla \cdot (u\theta)$. Therefore, by using integration by parts, the integral equation (3) is equivalent to

$$\theta(t) = \Theta(t) - B(\theta,\theta)(t), \tag{17}$$

where

$$\Theta(t) = G(t) * \theta_0, \quad B(v, w) := \int_0^t \nabla G(t-s,\cdot) * (\mathcal{R}v \cdot w)(s,\cdot)\, \mathrm{d}s.$$

The following lemma is a key step in this paper. Roughly speaking, the idea is that the smoothing effect should take place gradually in time. To deal with the bilinear term, we split the interval $(0, t)$ into $k + 1$ parts $I_0, \ldots, I_k$ evenly. In I_0, we put all the derivatives into the kernel $G(t-s,\cdot)$. We then shift the derivatives to $\mathcal{R}v \cdot w(s,\cdot)$ one by one as we move from I_0 to I_{k+1}. By doing this, one makes better use of the smoothing effect of the linear operator.

Lemma 3. *Assume that v and w are functions in $X^k_{q,T}$ for some $T \in (0,\infty]$, $q \in (\frac{2}{\gamma-1},\infty]$, and some integer $k \geqq 0$. Then $B(v,w)$ is also in $X^k_{q,T}$. For $k=0$, we have*

$$\|B(v,w)\|_{X^0_{q,T}} \le C_0 \|v\|^{1/2}_{X^0_{q,T}} \left\|t^\alpha v\right\|^{1/2}_{L^\infty_t L^q_x} \left\|t^\alpha w\right\|_{L^\infty_t L^q_x}, \tag{18}$$

and for $k>0$,

$$\begin{aligned}
\|B(v,w)\|_{Y^k_{q,T}} &\leqq C_0 \|v\|_{Y^k_{q,T}} \|t^\alpha w\|_{L^\infty_t((0,T))L^q_x} \\
&+ C_0 \|w\|_{Y^k_{q,T}} \|v\|^{1/2}_{X^0_{q,T}} \|t^\alpha v\|^{1/2}_{L^\infty_t((0,T))L^q_x} + C_1 \sum_{j=1}^{k-1} \binom{k}{j} \|v\|_{Y^j_{q,T}} \|w\|_{Y^{k-j}_{q,T}} \\
&+ \sum_{m=0}^{k-1}\sum_{j=0}^{m} \binom{m}{j} \cdot C_1^{k-m+1} k^{\frac{k}{\gamma}} m^{-\frac{m}{\gamma}} \|v\|_{Y^j_{q,T}} \|w\|_{Y^{m-j}_{q,T}},
\end{aligned} \tag{19}$$

where C_0 and C_1 are constants independent of k. Also in the case $m=0$, we define $0^0=1$.

Remark 7. We note that the existing methods of estimating the nonlinear term of the Navier–Stokes equations (see, for example, [1,3,14,15,17]) can also provide smoothness of the solution to (17). In those papers, the integrals corresponding to $B(v,w)$ are split into two parts for small time and large time, and with a suitable arrangement one can only obtain a less satisfactory estimate

$$\|\theta\|_{X^k_{q,T}} \leqq C^{k+1} k^{2k/\gamma}.$$

However, this does not imply the spatial analyticity of θ if $\gamma < 2$. This is the main reason why we shall use Lemma 3.

Proof of Lemma 3. For any function $v(t,x)$, let us write

$$\|v\|_{k,q,t} := \left\| t^{\frac{k}{\gamma}+1-\frac{1}{\gamma}-\frac{2}{\gamma q}} D^k_x v \right\|_{L^\infty_t((0,t))L^q_x}.$$

For any $q, q_1, q_2 \ge 1$ with $\frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}$, $0 \le \theta \le 1$, the following interpolation inequality obviously holds:

$$\|v\|_{k,q,t} \le \|v\|^\theta_{k,q_1,t} \|v\|^{1-\theta}_{k,q_2,t}. \tag{20}$$

We will use this fact later in this proof. Now by the Leibniz rule we have

$$\begin{aligned}
&D^k_x B(v,w) \\
&= \sum_{m=0}^{k} \int_{\frac{mt}{k+1}}^{\frac{(m+1)t}{k+1}} D^{k-m}_x \nabla_x G(t-s,\cdot) * \left(D^m_x(\mathcal{R}v(s)\cdot w(s))\right) \mathrm{d}s \\
&= \sum_{m=0}^{k}\sum_{j=0}^{m} \binom{m}{j} \int_{\frac{mt}{k+1}}^{\frac{(m+1)t}{k+1}} D^{k-m}_x \nabla_x G(t-s,\cdot) * (D^j_x \mathcal{R}v(s) \cdot D^{m-j}_x w(s))\, \mathrm{d}s.
\end{aligned}$$

Let $q_0', q_0'' \geq \frac{2}{\gamma-1}$. We shall take q_0 to be either $\frac{2}{\gamma-1}$ or q. The choice of q_0', q_0'' will be clear later. For now we shall derive a general estimate using q_0', q_0''. By Young's inequality and Hölder's inequality we have

$$\begin{aligned}\left\|D_x^k B(v,w)\right\|_{L_x^{q_0}} \leq \sum_{m=0}^{k}\sum_{j=0}^{m}\binom{m}{j}\int_{\frac{mt}{k+1}}^{\frac{(m+1)t}{k+1}} \left\|D_x^{k-m}\nabla_x G(t-s,\cdot)\right\|_{L_x^r} \\ \cdot\left\|\left(D_x^j \mathcal{R}v(s)\right\|_{L_x^{q_0'}}\left\|D_x^{m-j}w(s)\right)\right\|_{L_x^{q_0''}}\, \mathrm{d}s, \qquad (21)\end{aligned}$$

where r, q_0', q_0'' satisfy

$$1+\frac{1}{q_0} = \frac{1}{r}+\frac{1}{q_0'}+\frac{1}{q_0''}. \qquad (22)$$

We now estimate the right-hand side of (21). There are three cases.

Case 1. $1 \leq m \leq k-1$. In this case as we shall see, we need to require that $q_0', q_0'' \geq \frac{2}{\gamma-1}$ and $1-\frac{1}{r} \leq \frac{\gamma-1}{2}$. By (12) we have

$$\begin{aligned}&\int_{\frac{mt}{k+1}}^{\frac{(m+1)t}{k+1}} \left\|D_x^{k-m}\nabla_x G(t-s,\cdot)\right\|_{L_x^r}\left\|\left(D_x^j\mathcal{R}v(s)\right\|_{L_x^{q_0'}}\left\|D_x^{m-j}w(s)\right)\right\|_{L_x^{q_0''}}\,\mathrm{d}s \\ &\leq C_\gamma^{k-m+1}(k-m+1)^{\frac{k-m+1}{\gamma}}\int_{\frac{mt}{k+1}}^{\frac{(m+1)t}{k+1}}(t-s)^{-\frac{k-m+1}{\gamma}-\frac{2}{\gamma}\left(1-\frac{1}{r}\right)} \\ &\cdot s^{-\frac{m}{\gamma}-(2-\frac{2}{\gamma}-\frac{2}{\gamma q_0'}-\frac{2}{\gamma q_0''})}\,\mathrm{d}s\cdot\|\mathcal{R}v\|_{j,q_0',t}\,\|w\|_{m-j,q_0'',t} \\ &\leq t^{-\frac{k}{\gamma}-(1-\frac{1}{\gamma}-\frac{2}{\gamma q_0})}\cdot\beta(k,m)\cdot\|\mathcal{R}v\|_{j,q_0',t}\,\|w\|_{m-j,q_0'',t}\,,\end{aligned}$$

where $\beta(k,m)$ is bounded by

$$\begin{aligned}\beta(k,m) &\leq C_\gamma^{k-m+1}(k-m+1)^{\frac{k-m+1}{\gamma}}\int_{\frac{m}{k+1}}^{\frac{m+1}{k+1}}(1-s)^{-\frac{k-m+1}{\gamma}-\frac{2}{\gamma}\left(1-\frac{1}{r}\right)} \\ &\quad\cdot s^{-\frac{m}{\gamma}-\left(2-\frac{2}{\gamma}-\frac{2}{\gamma q_0'}-\frac{2}{\gamma q_0''}\right)}\,\mathrm{d}s \\ &\leq C_\gamma^{k-m+1}(k-m+1)^{\frac{k-m+1}{\gamma}}\cdot\frac{1}{k+1} \\ &\quad\cdot\left(1-\frac{m+1}{k+1}\right)^{-\frac{k-m+1}{\gamma}-\frac{2}{\gamma}\left(1-\frac{1}{r}\right)}\left(\frac{m}{k+1}\right)^{-\frac{m}{\gamma}-\left(2-\frac{2}{\gamma}-\frac{2}{\gamma q_0'}-\frac{2}{\gamma q_0''}\right)} \\ &\leq C_\gamma^{k-m+1}\cdot(2e)^{\frac{1}{\gamma}}\cdot(k+1)^{\frac{k}{\gamma}+1-\frac{1}{\gamma}-\frac{2}{\gamma q_0}}\cdot m^{-\frac{m}{\gamma}-\left(2-\frac{2}{\gamma}-\frac{2}{\gamma q_0'}-\frac{2}{\gamma q_0''}\right)}.\end{aligned}$$

Now if

$$1-\frac{1}{\gamma}-\frac{2}{\gamma q_0} \leq 2-\frac{2}{\gamma}-\frac{2}{\gamma q_0'}-\frac{2}{\gamma q_0''},$$

or by (22) equivalently,

$$1-\frac{1}{r}\le\frac{\gamma-1}{2},$$

then we have

$$\beta(k,m)\le C_\gamma^{k-m+1}\cdot(2e)^{\frac{1}{\gamma}}\cdot(k+1)^{\frac{k}{\gamma}+1-\frac{1}{\gamma}-\frac{2}{\gamma q_0}}\cdot m^{-\frac{m}{\gamma}-\left(1-\frac{1}{\gamma}-\frac{2}{\gamma q_0}\right)}.$$

If $1\le m\le\frac{k}{2}$, then clearly

$$\begin{aligned}\beta(k,m)&\le C_\gamma^{k-m+1}\cdot(k+1)\cdot(2e)^{\frac{1}{\gamma}}\frac{(k+1)^{\frac{k}{\gamma}}}{m^{\frac{m}{\gamma}}}\\&\le C_2^{k-m+1}\cdot k^{\frac{k}{\gamma}}m^{-\frac{m}{\gamma}},\end{aligned}$$

where C_2 is given by

$$C_2=C_\gamma\max_{k\ge0}(100(k+1))^{\frac{2}{k+1}}.$$

If $\frac{k}{2}\le m\le k-1$, then it is also not difficult to see that for some constant $C_3=100C_\gamma$,

$$\beta(k,m)\le C_3^{k-m+1}\frac{k^{\frac{k}{\gamma}}}{m^{\frac{m}{\gamma}}}.$$

Case 2. $m=0$. In this case, if $1<\gamma<2$, then we require only $q_0',q_0''\ge\frac{2}{\gamma-1}$. If $\gamma=2$, then q_0',q_0'' cannot both be ∞. We have by (12),

$$\begin{aligned}&\int_0^{\frac{t}{k+1}}\left\|D_x^k\nabla_xG(t-s,\cdot)\right\|_{L_x^r}\|\mathcal{R}v(s,\cdot)\|_{L_x^{q_0'}}\|w(s,\cdot)\|_{L_x^{q_0''}}\,\mathrm{d}s\\&\le t^{-\frac{k}{\gamma}-\left(1-\frac{1}{\gamma}-\frac{2}{\gamma q_0}\right)}C_\gamma^{k+1}(k+1)^{\frac{k+1}{\gamma}}\\&\cdot\int_0^{\frac{1}{k+1}}(1-s)^{-\frac{k+1}{\gamma}-\frac{2}{\gamma}\left(1-\frac{1}{r}\right)}s^{-\left(2-\frac{2}{\gamma}-\frac{2}{\gamma q_0'}-\frac{2}{\gamma q_0''}\right)}\mathrm{d}s\cdot\|\mathcal{R}v\|_{0,q_0',t}\|w\|_{0,q_0'',t}\\&\le t^{-\frac{k}{\gamma}-\left(1-\frac{1}{\gamma}-\frac{2}{\gamma q_0}\right)}C_4^{k+1}\cdot k^{\frac{k}{\gamma}}\int_0^{\frac{1}{2}}s^{-\left(2-\frac{2}{\gamma}-\frac{2}{\gamma q_0'}-\frac{2}{\gamma q_0''}\right)}\mathrm{d}s\cdot\|\mathcal{R}v\|_{0,q_0',t}\|w\|_{0,q_0'',t},\end{aligned}$$

where C_4 is some sufficiently large constant independent of k. Clearly in the last line, for the integral over s to converge, we must have the following condition on q_0',q_0''. If $1<\gamma<2$, then we require $q_0',q_0''\ge\frac{2}{\gamma-1}$. If $\gamma=2$, then q_0',q_0'' cannot both be ∞.

Case 3. $m = k$. In this case we require $q_0' \geq \frac{2}{\gamma-1}$, $q_0'' \geq \frac{2}{\gamma-1}$, and $1 - \frac{1}{r} < \frac{\gamma-1}{2}$. Note that r is not allowed to take the critical value $\frac{2}{3-\gamma}$. This will become clear in the following calculation. By (12), we have

$$\begin{aligned}
&\int_{1-\frac{t}{k+1}}^{t} \|\nabla_x G(t-s,\cdot)\|_{L_x^r} \left\| D_x^j \mathcal{R}v(s,\cdot)\right\|_{L_x^{q_0'}} \left\| D_x^{k-j} w(s,\cdot)\right\|_{L_x^{q_0''}} \mathrm{d}s \\
&\leq t^{-\frac{k}{\gamma}-\left(1-\frac{1}{\gamma}-\frac{2}{\gamma q_0}\right)} C_\gamma \cdot \|\mathcal{R}v\|_{j,q_0',t} \|w\|_{k-j,q_0'',t} \\
&\cdot \int_{1-\frac{1}{k+1}}^{1} (1-s)^{-\frac{1}{\gamma}-\frac{2}{\gamma}\left(1-\frac{1}{r}\right)} s^{-\frac{k}{\gamma}-\left(2-\frac{2}{\gamma}-\frac{2}{\gamma q_0}-\frac{2}{\gamma q_0'}\right)} \mathrm{d}s \\
&\leq t^{-\frac{k}{\gamma}-\left(1-\frac{1}{\gamma}-\frac{2}{\gamma q_0}\right)} \cdot \|\mathcal{R}v\|_{j,q_0',t} \|w\|_{k-j,q_0'',t} \cdot C_5 \cdot \int_{\frac{1}{2}}^{1} (1-s)^{-\frac{1}{\gamma}-\frac{2}{\gamma}\left(1-\frac{1}{r}\right)} \mathrm{d}s,
\end{aligned}$$

where C_5 is another constant which does not depend on k. Clearly for the integral over s in the last line above to converge, we must impose the condition $1-\frac{1}{r} < \frac{\gamma-1}{2}$.

From the analysis in the above three cases, we see that for any $q_j', q_j'' \geq \frac{2}{\gamma-1}$ we have

$$\begin{aligned}
&\|B(v,w)\|_{k,q_0,t} \\
&\leq A_1(r_1) \sum_{m=1}^{k-1} \sum_{j=0}^{m} \binom{m}{j} C_6^{k-m+1} k^{\frac{k}{\gamma}} m^{-\frac{m}{\gamma}} \|\mathcal{R}v\|_{j,q_1',t} \|w\|_{m-j,q_1'',t} \\
&\quad + A_2(q_2', q_2'', \gamma) \cdot C_6^{k+1} \cdot k^{\frac{k}{\gamma}} \|\mathcal{R}v\|_{0,q_2',t} \|w\|_{0,q_2'',t} \\
&\quad + A_3(r_3) \cdot C_6 \cdot \sum_{j=0}^{k} \binom{k}{j} \|\mathcal{R}v\|_{j,q_3',t} \|w\|_{k-j,q_3'',t}, \qquad (23)
\end{aligned}$$

where

$$1 + \frac{1}{q_0} = \frac{1}{r_j} + \frac{1}{q_j'} + \frac{1}{q_j''}, \quad j = 1,2,3 \qquad (24)$$

$$A_1(r_1) \begin{cases} \leq C_7, & \text{if } 1 - \frac{1}{r_1} \leq \frac{\gamma-1}{2}, \\ = \infty, & \text{otherwise} \end{cases}$$

$$A_2(q_2', q_2'', \gamma) \begin{cases} = \infty, & \text{if } \gamma = 2 \text{ and } q_2' = q_2'' = \infty, \\ \leq C_7, & \text{otherwise} \end{cases}$$

$$A_3(r_3) \begin{cases} \leq C_7, & \text{if } 1 - \frac{1}{r_3} < \frac{\gamma-1}{2}, \\ = \infty, & \text{otherwise,} \end{cases}$$

and C_6 and C_7 are sufficiently large constants independent of k. The first two terms on the right-hand side of (23) can be dealt with easily. We take $q_1'' = q_2'' = q_0$,

$q_1' = q_2' = \frac{2}{\gamma-1}$ and note that by this choice $A_1, A_2 \le C_7$. By the continuity of the Riesz operator $\mathcal{R}$ on L^p for $1 < p < \infty$, we have

$$\begin{aligned}\|B(v,w)\|_{k,q_0,t} &\le \sum_{m=0}^{k-1}\sum_{j=0}^{m}\binom{m}{j} C_8^{k-m+1} k^{\frac{k}{\gamma}} m^{-\frac{m}{\gamma}} \|v\|_{j,\frac{2}{\gamma-1},t}\,\|w\|_{m-j,q_0,t}\\ &\quad + A_3(r_3)\cdot C_6 \cdot \sum_{j=0}^{k}\binom{k}{j}\|\mathcal{R}v\|_{j,q_3',t}\,\|w\|_{k-j,q_3'',t}\\ &\le \sum_{m=0}^{k-1}\sum_{j=0}^{m}\binom{m}{j} C_8^{k-m+1} k^{\frac{k}{\gamma}} m^{-\frac{m}{\gamma}} \|v\|_{Y^j_{q,t}}\,\|w\|_{Y^{m-j}_{q,t}}\\ &\quad + A_3(r_3)\cdot C_6 \cdot \sum_{j=0}^{k}\binom{k}{j}\|\mathcal{R}v\|_{j,q_3',t}\,\|w\|_{k-j,q_3'',t}\,, \end{aligned} \tag{25}$$

where C_8 is another large constant independent of k. The last term on the right-hand side of (25) needs to be estimated more carefully. The problem is that we must require $1 - \frac{1}{r_3} < \frac{\gamma-1}{2}$ in order that $A_3(r_3)$ be finite. Note that we cannot take q_3' to be ∞ since the Riesz operator $\mathcal{R}$ is only continuous on L^p for $1 < p < \infty$. This fact motivates us to bound $\|\mathcal{R}v\|_{j,q_3',t}$ by $\|v\|_{j,q_3',t}$ and then interpolate between $\|v\|_{j,\frac{2}{\gamma-1},t}$ and $\|v\|_{j,q,t}$. More precisely, we take $q_3' \in (\frac{2}{\gamma-1}, \infty)$ such that

$$\frac{1}{q_3'} = \frac{\gamma-1}{4} + \frac{1}{2q}.$$

Since $1 < q_3' < \infty$, by the continuity of the Riesz operator $\mathcal{R}$ and (20), we have for some constant C_9,

$$\|\mathcal{R}v\|_{j,q_3',t} \le C_9\,\|v\|_{j,q_3',t} \le C_9\,\|v\|^{\frac12}_{j,\frac{2}{\gamma-1},t}\,\|v\|^{\frac12}_{j,q,t}\,.$$

To satisfy the condition $1 - \frac{1}{r_3} < \frac{\gamma-1}{2}$, we take $q_3'' = q_0$. By (24) we have

$$1 + \frac{1}{q_0} = \frac{1}{r_3} + \frac{1}{2q} + \frac{\gamma-1}{4} + \frac{1}{q_0}.$$

For either $q_0 = 2/(\gamma-1)$ or $q_0 = q > 2/(\gamma-1)$, we easily check that $1 - \frac{1}{r_3} < \frac{\gamma-1}{2}$ and therefore $A_3(r_3)$ is bounded. Now for $1 \le j \le k-1$, it is obvious that

$$\begin{aligned}\|\mathcal{R}v\|_{j,q_3',t}\,\|w\|_{k-j,q_3'',t} &\le C_9\,\|v\|^{\frac12}_{j,\frac{2}{\gamma-1},t}\,\|v\|^{\frac12}_{j,q,t}\,\|w\|_{k-j,q_0,t}\\ &\le C_9\,\|v\|_{Y^j_{q,t}}\,\|w\|_{Y^{k-j}_{q,t}}\,.\end{aligned}$$

For $j = 0$, we have

$$\begin{aligned}\|\mathcal{R}v\|_{0,q_3',t}\,\|w\|_{k,q_3'',t} &\le C_9\,\|v\|^{\frac12}_{0,\frac{2}{\gamma-1},t}\,\|v\|^{\frac12}_{0,q,t}\,\|w\|_{k,q_0,t}\\ &\le C_9\,\|v\|^{\frac12}_{X^0_{q,t}}\,\|v\|^{\frac12}_{0,q,t}\,\|w\|_{Y^k_{q,t}}.\end{aligned}$$

The case for $j = k$ follows similarly. Estimate (19) is now proved by taking C_0, C_1 to be sufficiently large but independent of k. The lemma is proved.

□

4. Existence of solutions in $X^k_{q,T}$

We proceed the proof of the existence of solutions in $X^k_{q,T}$ by using the bilinear estimate (3) and a contraction argument. Denote $\theta_1 = \Theta$. Let $\mathcal{F}$ be a map in $X^k_{q,T}$ defined as

$$\mathcal{F}(\theta) = \Theta - B(\theta, \theta).$$

We define inductively $\theta_{j+1} = \mathcal{F}(\theta_j)$.

Proposition 1. *Let $\gamma \in (1, 2]$, $\theta_0 \in L^{\frac{2}{\gamma-1}}$, and $q \in (\frac{2}{\gamma-1}, \infty]$. Then there exists $T > 0$ such that the initial value problem for (17) has a unique solution $\theta(t, x)$ in $X^k_{q,T}$ for any integer $k \geqq 0$.*

Proof. The well-posedness of (1) in $X^0_{q,T}$ for some $T > 0$ is a known result (See, for example, [1]). Here we give a sketched proof for the sake of completeness. *Step 1: Existence in $X^0_{q,T}$.* Due to Lemma 1,

$$\|\Theta\|_{X^0_{q,\infty}} \leqq R := C_\gamma \|\theta_0\|_{L^{2/(\gamma-1)}}$$

is bounded and

$$\lim_{t\to 0} t^\alpha \|\Theta\|_{L^q_x} = 0.$$

Let $\varepsilon \in (0, R)$ be a small positive number satisfying

$$4C_0 R^{1/2}\varepsilon^{3/2} < \varepsilon, \quad 2C_0(\varepsilon + (\varepsilon R)^{1/2}) < 1/6, \tag{26}$$

where C_0 is the constant in Lemma 3. We choose T sufficiently small such that

$$\|t^\alpha \Theta\|_{L^\infty_t((0,T))L^q_x} < \varepsilon.$$

Owing to (18), it is easy to check that for any $j \geqq 1$

$$\|t^\alpha \theta_j\|_{L^\infty_t((0,T))L^q_x} < 2\varepsilon, \quad \|\theta_j\|_{X^0_{q,T}} < 2R. \tag{27}$$

Since by (18), (27), and the second inequality in (26), for any $j \geqq 2$,

$$\begin{aligned}
\|\theta_{j+1} - \theta_j\|_{X^0_{q,T}} &\leqq \|B(\theta_j - \theta_{j-1}, \theta_j)\|_{X^0_{q,T}} + \|B(\theta_{j-1}, \theta_j - \theta_{j-1})\|_{X^0_{q,T}} \\
&\leqq C_0\|\theta_j - \theta_{j-1}\|_{X^0_{q,T}}(2\varepsilon + 2(\varepsilon R)^{1/2}) \\
&\leqq \|\theta_j - \theta_{j-1}\|_{X^0_{q,T}}/2.
\end{aligned}$$

Therefore,

$$\|\theta_{j+1} - \theta_j\|_{X^0_{q,T}} \leqq \bar{C} 2^{-j},$$

and the sequence $\{\theta_j\}$ is Cauchy in $X^0_{q,T}$. Thus, it converges to some $\theta \in X^0_{q,T}$, which is a fixed point of $\mathcal{F}$ and is a mild solution of (1) in $X^0_{q,T}$.

Step 2: Existence in $X^k_{q,T}$. Next, we show that $\{\theta_j\}$ is also a Cauchy sequence in $X^k_{q,T}$ for any $k \geqq 1$. Based on the linear estimate and the bilinear estimate which we obtained before, the argument is quite standard.

From the proof above it is clear that, for any $j \geqq 1$,

$$C_0 \|\theta_j\|^{1/2}_{X^0_{q,T}} \|t^\alpha \theta_j\|^{1/2}_{L^\infty_t((0,T))L^q_x} < 1/6. \tag{28}$$

By Lemma 1,

$$\|\Theta\|_{X^k_{q,\infty}} \leqq R_k := C_\gamma^{k+1} k^{k/\gamma} \|\theta_0\|_{L^{2/(\gamma-1)}}$$

is bounded for $k \geqq 1$. We first prove inductively on k that, for each $k \geqq 0$, $\|\theta_j\|_{X^k_{q,T}}$, $j \geqq 0$ are uniformly bounded. The case when $k = 0$ is proved in the first step. Now assume that we have

$$\|\theta_j\|_{X^k_{q,T}} \leqq C'_k, \quad \text{for } j \geqq 1.$$

This, together with (19), implies that $\|\theta_{j+1}\|_{X^{k+1}_{q,T}}$ is less than

$$\begin{aligned}
&R_{k+1} + 2C_0 \|\theta_j\|_{X^{k+1}_{q,T}} \|\theta_j\|^{1/2}_{X^0_{q,T}} \|t^\alpha \theta_j\|^{1/2}_{L^\infty_t((0,T))L^q_x} \\
&\quad + C_1 \sum_{l=1}^{k-1} \binom{k}{l} \|\theta_j\|_{Y^l_{q,T}} \|\theta_j\|_{Y^{k-l}_{q,T}} \\
&\quad + \sum_{m=0}^{k-1} \sum_{l=0}^{m} \binom{m}{l} \cdot C_1^{k-m+1} k^{\frac{k}{\gamma}} m^{-\frac{m}{\gamma}} \|\theta_j\|_{Y^l_{q,T}} \|\theta_j\|_{Y^{m-l}_{q,T}} \\
&\leqq R_{k+1} + \frac{1}{3} \|\theta_j\|_{X^{k+1}_{q,T}} + C''_k (C'_k)^2,
\end{aligned}$$

where C''_k is a constant depending only on k. Then by iteration it follows immediately that

$$\|\theta_j\|_{X^k_{q,T}} \leqq C''_{k+1} := \frac{3}{2}(R_{k+1} + C''_k (C'_k)^2), \tag{29}$$

which shows that, for each k, $\|\theta_j\|_{X^k_{q,T}}$, $j \geqq 1$ are uniformly bounded.

Next we verify that $\{\theta_j\}$ is a Cauchy sequence in $X^k_{q,T}$:

$$\|\theta_{j+1} - \theta_j\|_{X^k_{q,T}} \leqq \bar{C}_k 2^{-j}. \tag{30}$$

Again we proceed by an induction on k. The case when $k = 0$ is proved in step 1. Assume this is already proved for $l = 1, 2, \ldots, k$. Since

$$\theta_{j+1} - \theta_j = B(\theta_j - \theta_{j-1}, \theta_j) + B(\theta_{j-1}, \theta_j - \theta_{j-1}),$$

(19), our inductive assumption, and (29) yield

$$\|\theta_{j+1} - \theta_j\|_{X^k_{q,T}} \leqq \frac{1}{3}\|\theta_j - \theta_{j-1}\|_{X^k_{q,T}} + C''_k 2^{-j}.$$

From this and (29), we easily conclude (30) for some $\bar{C}_k$ depending only on k. As a consequence, $\{\theta_j\}$ is Cauchy in $X^k_{q,T}$ and it converges to some $\theta \in X^k_{q,T}$, which is a fixed point of $\mathcal{F}$ and a solution of (1) in $X^k_{q,T}$. The proposition is proved. □

5. Estimate of analyticity rate

The next lemma is quite elementary.

Lemma 4. *Suppose C_2, C'_2, A_0, and A_1 are positive numbers, and $A_k, k \geqq 2$ satisfies*

$$A_k = C_2 + C'_2 \sum_{0 \leqq i \leqq j \leqq k-1, i+j \leqq k} A_i A_j. \tag{31}$$

Then there exists C_3 depending only on A_0, A_1, C_2, C'_2 such that $A_k \leqq C_3^{k+1}$.

Proof. Without loss of generality we can assume that $A_0 = A_0 = C_2$. By considering $C'_2 A_k$ instead of A_k, we may further assume $C'_2 = 1$. We define a generating function $f(x) = \sum_{i=0}^{\infty} A_0 x^k$. Due to (31), $f(x)$ satisfies

$$(2C_2 + 1) f(x) = \frac{C_2 + (f(x))^2}{1 - x} + C_2^2(1 - x).$$

This gives

$$f(x) = \frac{1}{2}\left[(2C_2 + 1)(1 - x) - \sqrt{(1 - x)^2(1 + 4C_2) - 4C_2}\right].$$

Notice that the coefficients of $f(x)$ grow at most exponentially and satisfy the recursive formula (31). Therefore, we can find $C_3 > 0$ such that $A_k \leqq C_3^{k+1}$. □

Now we are ready to estimate the $X^k_{q,T}$ norms of θ.

Proposition 2. *There exists a constant C independent of k satisfying*

$$\|\theta\|_{X^k_{q,T}} \leqq C^{k+1} k^k. \tag{32}$$

Proof. Due to (17), we have

$$\|\theta\|_{X^k_{q,T}} \leqq \|\Theta\|_{X^k_{q,T}} + \|B(\theta,\theta)\|_{X^k_{q,T}}.$$

Now we use (13), (19), and (28) and replace C_1 with $\max\{C_\gamma, C_1\}$ to obtain

$$\|\theta\|_{X^k_{q,T}} \leqq C_1^{k+1} k^{\frac{k}{\gamma}} + \frac{1}{3}\|\theta\|_{X^k_{q,T}} + C_1 \sum_{j=1}^{k-1} \binom{k}{j} \|\theta\|_{X^j_{q,T}} \|\theta\|_{X^{k-j}_{q,T}}$$
$$+ \sum_{m=0}^{k-1} \sum_{j=0}^{m} \binom{m}{j} \cdot C_1^{k-m+1} k^{\frac{k}{\gamma}} m^{-\frac{m}{\gamma}} \|\theta\|_{X^j_{q,T}} \|\theta\|_{X^{m-j}_{q,T}}.$$

Thus,

$$\|\theta\|_{X^k_{q,T}} \leqq \frac{3}{2}\left[C_1^{k+1} k^k + C_1 \sum_{j=1}^{k-1} \binom{k}{j} \|\theta\|_{X^j_{q,T}} \|\theta\|_{X^{k-j}_{q,T}} \right.$$
$$\left. + \sum_{m=0}^{k-1} \sum_{j=0}^{m} \binom{m}{j} \cdot C_1^{k-m+1} k^k m^{-m} \|\theta\|_{X^j_{q,T}} \|\theta\|_{X^{m-j}_{q,T}}. \right] \tag{33}$$

Finally, we denote $A_k = \|\theta\|_{X^k_{q,T}} C_1^{-k} k^{-k}$ and apply Lemma 4 to get

$$A_k \leqq C_3^k, \quad \|\theta\|_{X^k_{q,T}} \leqq C_3^{k+1} C_1^k k^k.$$

Here we also use the obvious inequality $\binom{m}{j} \leqq m^m j^{-j}(m-j)^{-m+j}$. This completes the proof of Proposition 2. □

6. Proofs of Theorem 1 and 2

We are ready to prove Theorems 1 and 2, and Corollary 1.

Proof of Theorem 1. This follows directly from Propositions 1 and 2. □

Proof of Corollary 1. Since uniqueness follows in a standard way from the local uniqueness result, we only have to show the existence of a mild solution in $X^k_{q,T}$ for any $T > 0$. From Theorem 1, there exists a mild solution $\theta \in X^k_{q,T_1}$ for some $T_1 > 0$. From the proof of Proposition 1, one can see that the only conditions needed are the boundedness of $\|\Theta\|_{X^k_{q,T}}$ and the smallness of $t^\alpha \|\Theta\|_{L^q_x}$. By the linear estimate in Section 2, the boundedness of $\|\Theta\|_{X^k_{q,T}}$ can be deduced from the boundedness of $\|\theta_0\|_{L^{2/(\gamma-1)}}$. The smallness of $t^\alpha \|\Theta\|_{L^q_x}$ can be deduced from either the boundedness of $\|\theta_0\|_{L^{2/(\gamma-1)}}$, or the boundedness of $\|\theta_0\|_{L^q}$ if we already know that θ_0 is in L^q. □

If T_1 is finite, by the L^p maximum principle, we have for any $s \in (T_1/2, T_1)$

$$\|\theta(s,\cdot)\|_{L^q_x} \leqq \|\theta(T_1/2,\cdot)\|_{L^q_x}, \quad \|\theta(s,\cdot)\|_{L_x^{2/(\gamma-1)}} \leqq \|\theta_0\|_{L_x^{2/(\gamma-1)}}. \tag{34}$$

For each $s \in (T_1/2, T_1)$, we solve (1) with initial data $\theta(s,\cdot)$ in place of θ_0. Due to (34), there exists a $\delta > 0$ uniformly with respect to s such that (1) has a mild solution for $t \in [s, s+\delta)$. Letting $s \to T_1$ gives a solution in $X^k_{q,T_1+\delta}$. We can iterate this process and obtain a solution in $X^k_{q,T}$.

This argument together with (4) also implies (5). Indeed, by Theorem 1, (5) holds for $t \in (0, T_1]$. For $t \in [T_1, T)$, we can consider $\theta(t,\cdot)$ as a solution of (1) starting from $s := t - \min\{\delta, T_1\}/2$ with initial data $\theta(s,\cdot)$. The proof of the corollary is then completed.

Proof of Theorem 2. Due to Corollary 1, we only have to prove (6). The following lemma is proved in a recent preprint of CARRILO and FERREIRA [1]. □

Lemma 5. *Let $q \in (2/(\gamma-1), \infty]$ and $\theta \in X^0_{q,\infty}$ be the mild solution of (1) with initial data $\theta_0 \in L^{2/(\gamma-1)}$. Then we have*

$$\|t^\alpha \theta\|_{L_t^\infty((0,\infty))L^q_x} \leqq C'\|\theta_0\|_{L^{2/(\gamma-1)}}, \tag{35}$$

for some positive constant C' depending only on γ.

Let ε and R be the constants in the proof of Proposition 1. We choose δ sufficiently small such that

$$(2\delta)^\alpha(1-\delta)^{-\alpha}C'\|\theta_0\|_{L^{2/(\gamma-1)}} \leqq \varepsilon. \tag{36}$$

For any fixed $t_1 > 0$, let $s = (1-\delta)t_1$. Let $\bar\theta \in X^0_{q,\infty}$ be the unique mild solution of (1) with initial data $\bar\theta_0 = \theta(s,\cdot)$. Denote $\bar\Theta$ to be the corresponding solution to the linear equation. Then by Lemma 1 and the L^p maximum principle, we have for any $k \geqq 0$

$$\|\bar\Theta\|_{X^0_{q,\infty}} \leqq R, \quad \|\bar\Theta\|_{X^k_{q,\infty}} \leqq R_k,$$

where R and R_k are the same constants as in the proof of Proposition 1. From (35) and (36), we have

$$\begin{aligned}\|t^\alpha\bar\Theta\|_{L_t^\infty((0,2\delta t_1))L^q_x} &\leqq (2\delta t_1)^\alpha\|\bar\Theta\|_{L_t^\infty((0,2\delta t_1))L^q_x} \leqq (2\delta t_1)^\alpha\|\theta(s)\|_{L^q_x}\\ &\leqq (2\delta t_1)^\alpha s^{-\alpha}C'\|\theta_0\|_{L^{2/(\gamma-1)}} \leqq \varepsilon.\end{aligned}$$

From the proofs of Proposition 1 and 2, it holds that

$$\|\bar\theta\|_{X^k_{q,2\delta t_1}} \leqq C^{k+1}k^k, \tag{37}$$

for any $k \geqq 0$ and some constant C independent of k and t_1. Now by the uniqueness of the mild solution, we have $\bar\theta(t) = \theta(t+s)$ for any $t \geqq 0$. Therefore, for any integer $k \geqq 0$,

$$t_1^{\frac{k}{\gamma}}\|D^k\theta(t_1,\cdot)\|_{L_x^{\frac{2}{\gamma-1}}} = t_1^{\frac{k}{\gamma}}\|D^k\bar\theta(\delta t_1,\cdot)\|_{L_x^{\frac{2}{\gamma-1}}} \leqq \delta^{-\frac{k}{\gamma}}C^{k+1}k^k, \tag{38}$$

$$t_1^{\frac{k}{\gamma}+\alpha}\|D^k\theta(t_1,\cdot)\|_{L_x^q} = t_1^{\frac{k}{\gamma}+\alpha}\|D^k\bar{\theta}(\delta t_1,\cdot)\|_{L_x^q} \leqq \delta^{-\frac{k}{\gamma}-\alpha}C^{k+1}k^k. \tag{39}$$

Since $t_1 > 0$ is arbitrary, (38) and (39) imply (6) with a different constant C. The theorem is proved.

7. A second proof of Theorem 2: fractional bootstrapping

This section is devoted to a second proof of Theorem 2. Here we use the idea of the fractional bootstrapping by noticing that in the subcritical case we can put a little bit more than one derivative in the kernel and still keep the convergence of the integral of the bilinear term. The advantage is that we can avoid using the contraction argument in $X^k_{q,T}$ for $k \geqq 1$ as long as the solution is known to be in $X^0_{q,T}$. Furthermore, the estimate of fractional derivatives can also be obtained almost immediately. In the first subsection, we illustrate this idea by proving the simpler case when $q < \infty$. After that, we prove the case $q = \infty$.

7.1. First case: $q < \infty$.

Fix q such that $2/(\gamma-1) < q < \infty$. Let $N \geq 1$ be an integer sufficiently large such that

$$1 + \frac{1}{N} + \frac{2}{q} < \gamma. \tag{40}$$

This assumption will be needed later (see (43) and (44)). For $k \geq 0, l = 0, 1, \ldots, N$, write

$$A(k,l) = \left\| t^{\frac{k+l/N}{\gamma}+\alpha} D^k \Lambda^{\frac{l}{N}}\theta \right\|_{L_t^\infty(0,\infty)L_x^q},$$

where $\alpha = 1 - \frac{1}{\gamma} - \frac{2}{q\gamma}$. We shall derive a set of recurrent inequalities for $A(k,l)$. The desired estimates on $A(k,l)$ follows from studying the recurrent relations. To this end, write also

$$B(k,l) = \left\| t^{\frac{k+l/N}{\gamma}+\alpha} D^k \Lambda^{\frac{l}{N}}\Theta \right\|_{L_t^\infty(0,\infty)L_x^q}.$$

Lemma 6. *For $1 \leq l \leq N$, we have for some constant $C > 0$ independent of k,*

$$\begin{aligned} A(k,l) \leq{}& B(k,l) + \beta(k,0)A(0,0)^2 \\ &+ \sum_{m=1}^{k}\sum_{j=0}^{m}\binom{m}{j}\beta(k,m)A(j,l-1)A(m-j,0) \end{aligned} \tag{41}$$

where for any $k \geq 0$, $0 \leq m \leq k$,

$$\beta(k,m) \leq C^{k-m+1}k^{\frac{k}{\gamma}} \cdot m^{-\frac{m}{\gamma}}.$$

For $l = 0$ *and* $k \geq 0$ *we have*

$$A(k+1, 0) \leq CA(k, N). \tag{42}$$

Proof. In this proof we shall denote by C the constants which may vary from line to line but do not depend on k. From (17), for $1 \leq l \leq N$, we have

$$\begin{aligned} D^k \Lambda^{\frac{l}{N}} \theta(t) = D^k \Lambda^{\frac{l}{N}} \Theta(t) &+ \int_0^{\frac{t}{k+1}} D_x^{k+1} \Lambda^{\frac{l}{N}} G(t-s, \cdot) * (\mathcal{R}\theta(s) \cdot \theta(s))\, \mathrm{d}s \\ &+ \sum_{m=1}^{k} \sum_{j=0}^{m} \binom{m}{j} \int_{\frac{mt}{k+1}}^{\frac{(m+1)t}{k+1}} D_x^{k-m+1} \Lambda^{\frac{1}{N}} G(t-s, \cdot) \\ &* \left(\Lambda^{\frac{l-1}{N}} (D_x^j (\mathcal{R}\theta(s)) \cdot D_x^{m-j} \theta(s)) \right) \mathrm{d}s. \end{aligned}$$

By Young's inequality and Hölder's inequality, we have

$$\begin{aligned} \left\| D^k \Lambda^{\frac{l}{N}} \theta(t) \right\|_{L_x^q} \leq \left\| D^k \Lambda^{\frac{l}{N}} \Theta(t) \right\|_{L_x^q} &+ \int_0^{\frac{t}{k+1}} \left\| D_x^{k+1} \Lambda^{\frac{l}{N}} G(t-s, \cdot) \right\|_{L_x^{\frac{q}{q-1}}} \\ &\cdot (\|\mathcal{R}\theta(s)\|_{L_x^q} \|\theta(s)\|_{L_x^q})\, \mathrm{d}s \\ &+ \sum_{m=1}^{k} \sum_{j=0}^{m} \binom{m}{j} \int_{\frac{mt}{k+1}}^{\frac{(m+1)t}{k+1}} \left\| D_x^{k-m+1} \Lambda^{\frac{1}{N}} G(t-s, \cdot) \right\|_{L_x^{\frac{q}{q-1}}} \\ &\cdot \left\| \Lambda^{\frac{l-1}{N}} (D_x^j (\mathcal{R}\theta(s)) \cdot D_x^{m-j} \theta(s)) \right\|_{L_x^{\frac{q}{2}}} \mathrm{d}s. \end{aligned}$$

To bound the last term in the above inequality, we use the following inequality for fractional derivatives,

$$\left\| \Lambda^{\varepsilon} (fg) \right\|_{L_x^{q/2}} \leq C \left(\left\| \Lambda^{\varepsilon} f \right\|_{L_x^q} \|g\|_{L_x^q} + \|f\|_{L_x^q} \left\| \Lambda^{\varepsilon} g \right\|_{L_x^q} \right).$$

This immediately gives us

$$\begin{aligned} A(k, l) \leq B(k, l) &+ \beta(k, 0) A(0, 0)^2 \\ &+ \sum_{m=1}^{k} \sum_{j=0}^{m} \binom{m}{j} \beta(k, m) A(j, l-1) A(m-j, 0), \end{aligned}$$

where $\beta(k, m)$ will be estimated below. We have three cases.

Case 1. estimate of $\beta(k, 0)$. By (16) we have

$$\begin{aligned} \beta(k, 0) &\leq C^{k+1} k^{\frac{k}{\gamma}} \int_0^{\frac{1}{k+1}} (1-s)^{-\frac{k+1+\frac{l}{N}}{\gamma} - \frac{2}{\gamma q}} s^{-2\alpha}\, \mathrm{d}s \\ &\leq C^{k+1} k^{\frac{k}{\gamma}} \int_0^{1/2} s^{-2\alpha}\, \mathrm{d}s \leq C^{k+1} k^{\frac{k}{\gamma}}. \end{aligned}$$

Note that the last integral converges since we have $q < \infty$.

Case 2. estimate of $\beta(k,m)$, $1 \le m \le k-1$. By (16) we have

$$\beta(k,m) \le C^{k-m+1} \cdot (k-m+1)^{\frac{k-m+1}{\gamma}} \cdot \int_{\frac{m}{k+1}}^{\frac{m+1}{k+1}} (1-s)^{-\frac{k-m+1+\frac{1}{N}}{\gamma} - \frac{2}{q\gamma}} s^{-\frac{m}{\gamma} - \frac{l-1}{N\gamma} - 2\alpha} \, \mathrm{d}s$$

$$\le C^{k-m+1}(k-m+1)^{\frac{k-m+1}{\gamma}} \left(1 - \frac{m+1}{k+1}\right)^{-\frac{k-m+1+\frac{1}{N}}{\gamma} - \frac{2}{q\gamma}} \cdot \left(\frac{m}{k+1}\right)^{-\frac{m}{\gamma} - \frac{l-1}{N\gamma} - 2\alpha} \cdot \frac{1}{k+1}$$

$$\le C^{k-m+1} k^{\frac{k}{\gamma}} \cdot m^{-\frac{m}{\gamma}} \cdot k^{\frac{l}{N\gamma} + \alpha} \cdot m^{-\frac{l-1}{N\gamma} - 2\alpha}.$$

Now if $m \le k/2$, then

$$\beta(k,m) \le C^{k-m+1} \cdot k^{\frac{k}{\gamma}} \cdot m^{-\frac{m}{\gamma}} \cdot k^{\frac{l}{N\gamma} + \alpha} \le (2C)^{k-m+1} \cdot k^{k/\gamma} \cdot m^{-\frac{m}{\gamma}}.$$

If $m > k/2$, then

$$\begin{aligned} \beta(k,m) &\le C^{k-m+1} \cdot k^{\frac{k}{\gamma}} \cdot m^{-\frac{m}{\gamma}} \cdot 2^{\frac{l-1}{N\gamma} + 2\alpha} k^{\frac{1}{N\gamma} - \alpha} \\ &\le (4^{1+\alpha} C)^{k-m+1} \cdot k^{\frac{k}{\gamma}} \cdot m^{-\frac{m}{\gamma}}. \end{aligned} \tag{43}$$

In the last inequality we have used the fact that $\frac{1}{N\gamma} - \alpha < 0$ (see (40)). Now, replacing C by $\max\{2C, 4^{\alpha+1}C\}$, we get

$$\beta(k,m) \le C^{k-m+1} k^{\frac{k}{\gamma}}.$$

Case 3. estimate of $\beta(k,k)$. Again by (16) we have

$$\begin{aligned} \beta(k,k) &\le C \int_{1-\frac{1}{k+1}}^{1} (1-s)^{-\frac{1+\frac{1}{N}}{\gamma} - \frac{2}{q\gamma}} s^{-\frac{k}{\gamma} - \frac{l-1}{N\gamma} - 2\alpha} \, \mathrm{d}s \\ &\le C \int_{1/2}^{1} (1-s)^{-\frac{1+\frac{1}{N}+\frac{2}{q}}{\gamma}} \, \mathrm{d}s \le C. \end{aligned} \tag{44}$$

By (40) the last integral converges.

The inequality (42) is due to the simple fact that the operator $\Lambda^{-1}\nabla$ maps L_x^q to L_x^q boundedly for any $1 < q < \infty$. The lemma is proved.

□

Lemma 7. *Let* $1 \le l \le N$, $k \ge 0$. *Let* $n = Nk + l$, $n_1 = Nj + l - 1$, $n_2 = N(m-j)$, *where* $0 \le j \le m$ *and* $1 \le m \le k$. *Then there exists a constant* C *depending only on* N *such that*

$$\binom{m}{j} \cdot k^k \cdot m^{-m} \le \left(\frac{n^n}{n_1^{n_1} \cdot n_2^{n_2}}\right)^{1/N}.$$

Proof. We compute

$$\binom{m}{j}\cdot k^k\cdot m^{-m}\le \frac{m^m}{j^j(m-j)^{m-j}}\frac{k^k}{m^m}\le \frac{\left(\frac{n-l}{N}\right)^{\frac{n-l}{N}}}{\left(\frac{n_2}{N}\right)^{\frac{n_2}{N}}\cdot\left(\frac{n_1-l+1}{N}\right)^{\frac{n_1-l+1}{N}}}$$

$$\le C\left(\frac{n^n}{n_1^{n_1}\cdot n_2^{n_2}}\right)^{1/N}.$$

The lemma is proved. □

Theorem 4. *There exists a constant $C>0$ such that*

$$A(k,l)\le C^{k+1}k^k,\quad \forall\, k\ge 0,\quad 0\le l\le N.$$

Proof. By Lemma 7 and replacing C by yet another large constant, it is easy to see that $A(k,l)\le F(n)$, where $n=Nk+l$ and $F(n)$ satisfies:

$$F(n)=C^n n^{\frac{n}{N}}+\sum_{n_1+n_2\le n-1}C^{n-(n_1+n_2)+1}\left(\frac{n^n}{n_1^{n_1}\cdot n_2^{n_2}}\right)^{\frac{1}{N}}F(n_1)F(n_2).$$

By Lemma 4 the sequence $F(n)n^{-n/N}C^{-n}$ grows at most exponentially. The theorem is proved. □

A quick inspection of our proof reveals that we can estimate any fractional derivatives. Indeed we shall prove a slightly more general result as follows (see Remark 4).

Proposition 3. *Let $q\in(\frac{2}{\gamma-1},\infty)$. For any integer $k\ge 0$ and any real number $\beta\ge 0$, we have*

$$\|D_x^k\Lambda^\beta\theta(t,\cdot)\|_{L_x^q}\le C^{k+\beta+1}t^{-\frac{k+\beta}{\gamma}-\alpha}\Gamma(k+\beta+1),$$

where $\alpha=1-\frac{1}{\gamma}-\frac{2}{q\gamma}$.

Proof. Take $0<\varepsilon\leqq 1/N$, where N is defined in (40). By using the same bilinear estimates as before, it is not difficult to show that

$$\|D_x^k\Lambda^\varepsilon\theta(t,\cdot)\|_{L_x^q}\le C^{k+\varepsilon+1}t^{-\frac{k+\varepsilon}{\gamma}-\alpha}\Gamma(k+\varepsilon+1),$$

where the constant C can be taken to be independent of ε. Now by the same bootstrapping argument as in Lemma 6 we get the estimate for $\beta\in(0,2)$ (one only needs to bootstrap at most $2N$ times). Finally use the fact that $\Lambda^{2m}=(-1)^m\Delta$ for any integer $m\ge 0$. □

7.2. Second case: $q = \infty$

As we shall show, this case follows easily from using the estimates of the case $q < \infty$. The following lemma is almost immediately obvious.

Lemma 8. *For any $m \geq 0$ and any $p \in (\frac{1}{\gamma-1}, \infty)$ we have*

$$\|D_x^m(u(t,\cdot)\cdot\theta(t,\cdot))\|_{L_x^p} \leq C^{m+1}\cdot m^m\cdot t^{-\frac{m}{\gamma}-\left(2-\frac{2}{\gamma}-\frac{2}{p\gamma}\right)}.$$

Proof. The estimate for $D^k u(t,\cdot)$ follows from the continuity of the Riesz operator $\mathcal{R}$. The result follows from the use of the Leibniz rule. $\square$

Theorem 5 (Rate of analyticity). *There exists a constant $C > 0$ such that*

$$\|t^{\frac{k}{\gamma}+\alpha}D_x^k\theta\|_{L_t^\infty L_x^\infty} \leq C^{k+1}k^k,$$

where $\alpha = 1-\frac{1}{\gamma}$. Consequently $\theta(t,\cdot)$ is analytic.

Proof. By (17) we have

$$\begin{aligned} D_x^k\theta(t) &= D_x^k\Theta(t) + \int_0^{\frac{t}{k+1}} D_x^{k+1}G(t-s,\cdot)*(u(s)\cdot\theta(s))\,\mathrm{d}s \\ &\quad + \sum_{m=1}^{k}\int_{\frac{mt}{k+1}}^{\frac{(m+1)t}{k+1}} D_x^{k-m+1}G(t-s,\cdot)*(D_x^m(u(s)\cdot\theta(s)))\,\mathrm{d}s. \end{aligned}$$

Fix p such that $p \in (\frac{2}{\gamma-1},\infty)$. Let $p' = \frac{p}{p-1}$. Then we have

$$\begin{aligned} &\|D_x^k\theta(t)\|_{L_x^\infty} \\ &\quad \leq \|D_x^k\Theta(t)\|_{L_x^\infty} + \int_0^{\frac{t}{k+1}} \|D_x^{k+1}G(t-s,\cdot)\|_{L_x^{p'}}\|u(s)\cdot\theta(s)\|_{L_x^p}\,\mathrm{d}s \\ &\qquad + \sum_{m=1}^{k}\int_{\frac{mt}{k+1}}^{\frac{(m+1)t}{k+1}} \|D_x^{k-m+1}G(t-s,\cdot)\|_{L_x^{p'}}\|D_x^m(u(s)\cdot\theta(s))\|_{L_x^p}\,\mathrm{d}s. \end{aligned}$$

By Lemma 8, the linear estimates on $\Theta(t)$, and the kernel G (see Lemma 1) we have

$$
\begin{aligned}
&\|t^{\frac{k}{\gamma}+\alpha} D_x^k \theta(t)\|_{L_t^\infty L_x^\infty} \\
&\le C^{k+1}k^k + C^{k+1}(k+1)^{\frac{k+1}{\gamma}} \int_0^{\frac{1}{k+1}} (1-s)^{-\frac{k+1}{\gamma}-\frac{2}{\gamma p}} s^{-\left(2-\frac{2}{\gamma}-\frac{2}{p\gamma}\right)}\,\mathrm{d}s \\
&+C^{k+1}\sum_{m=1}^{k} \int_{\frac{m}{k+1}}^{\frac{m+1}{k+1}} (1-s)^{-\frac{k-m+1}{\gamma}-\frac{2}{\gamma p}} \cdot (k-m+1)^{\frac{k-m+1}{\gamma}} \\
&\cdot m^m s^{-\frac{m}{\gamma}-\left(2-\frac{2}{\gamma}-\frac{2}{p\gamma}\right)}\,\mathrm{d}s \\
&\le C^{k+1}k^k + C^{k+1}\sum_{m=1}^{k-1}(k-m+1)^{\frac{k-m+1}{\gamma}} \cdot m^m \\
&\cdot \left(1-\frac{m+1}{k+1}\right)^{-\frac{k-m+1}{\gamma}-\frac{2}{\gamma p}} \cdot \left(\frac{m}{k+1}\right)^{-\frac{m}{\gamma}-\left(2-\frac{2}{\gamma}-\frac{2}{p\gamma}\right)} \cdot \frac{1}{k+1} \\
&\le C^{k+1}k^k + C^{k+1}(k+1)^{\frac{k}{\gamma}} \cdot m^{m\left(1-\frac{1}{\gamma}\right)} \\
&\le C^{k+1}k^k.
\end{aligned}
$$

□

8. Temporal Gevrey regularity of $\theta(t, x)$

In Sections 6 and 7 we have shown that, for any $t > 0$, there exists a constant C independent of k such that

$$\left\| D^k u(t,\cdot) \right\|_{L_x^q} \le C^{k+1} t^{-\frac{k}{\gamma}-\alpha} k^k, \tag{45}$$

where $\alpha = 1 - \frac{1}{\gamma} - \frac{2}{q\gamma}$. We now want to understand the temporal regularity of $\theta(t, x)$. To see that $\theta(t, x)$ is smooth (infinitely differentiable) in t for $0 < t < T$, we write (1) as:

$$\theta_t = -u\nabla\theta - (-\Delta)^{\gamma/2}\theta. \tag{46}$$

It is clear that by this relation higher-order derivatives of θ in t can be expressed in and are therefore bounded by terms only involving derivatives of θ in x. For this reason we expect that θ is smooth in t. It is also instructive to see how to get a reasonable bound on $\partial_t^m\theta$. If we ignore the nonlinear term $u\nabla\theta$, then heuristically speaking we have $\partial_t^m\theta \approx (-\Delta)^{m\gamma/2}\theta$. Therefore by (45) the best bound we can hope for is

$$\left\| \partial_t^m \partial_x^k \theta \right\|_{L_x^\infty} \leqq C^{m\gamma+k+1} \cdot t^{-\frac{k}{\gamma}-m-\left(1-\frac{1}{\gamma}\right)} \cdot (m\gamma+k)^{m\gamma+k}. \tag{47}$$

We conjecture that for small t the above bound is optimal and gives the correct asymptotics for the norms of high order t-derivatives of θ. It turns out (perhaps surprisingly) that the nonlinear term $u \cdot \nabla\theta$ does not spoil the conjectured bound

(47) too much. Indeed we shall prove a slightly less optimal bound on the mixed space–time derivatives which has the following form for small t:

$$\left\|\partial_t^m \partial_x^k \theta\right\|_{L_x^\infty} \leqq C^{2m+k+1} \cdot t^{-\frac{2m+k+1}{\gamma}} \cdot (2m+k)^{2m+k}. \tag{48}$$

Before we give the proof of Theorem 3, we shall explain the basic strategy of our method of proof. There are two main difficulties in estimating $\left\|\partial_t^m \partial_x^k \theta\right\|_{L_x^\infty}$. One is that we cannot estimate L_x^∞ directly using (46) since $u = \mathcal{R}\theta$ and the Riesz operator $\mathcal{R}$ only maps L^∞ to BMO (functions with bounded mean oscillation). We circumvent this difficulty by estimating instead $\left\|\partial_t^m \partial_x^k \theta\right\|_{L_x^q}$ for any fixed $q \in (\frac{2}{\gamma-1}, \infty)$ and then use the Sobolev imbedding theorem. The other difficulty arises from the fact that we must deal with the nonlocal operator $(-\Delta)^{\gamma/2}$ when $1 < \gamma < 2$. In order to avoid the use of the fractional Leibniz rule, we use the following interpolation inequality:

$$\left\|(-\Delta)^{\gamma/2}\phi\right\|_{L_x^q} \leq C(q)\, \|\phi\|_{L_x^q}^{1-\gamma/2} \,\|\Delta\phi\|_{L_x^q}^{\gamma/2}, \tag{49}$$

where $C(q)$ is some constant depending only on q. Let us first consider the regime $0 < t \leq 1$. Fix $q \in (\frac{2}{\gamma-1}, \infty)$ and let

$$C(k,m) = \left\| t^{\frac{2m+k+1}{\gamma}} \partial_t^m \partial_x^k \theta \right\|_{L_t^\infty(0,\infty) L_x^q}.$$

Remark that the scaling of the exponent of t is not optimal in the regime $0 < t \leq 1$. We have the following lemma concerning the growth of $C(k,m)$.

Lemma 9. *For any $k \geq 0$, $m \geq 0$, we have*

$$C(k,m) \leq C^{2m+k+1} \cdot (2m+k)^{2m+k}. \tag{50}$$

Proof. In this proof we denote by C_1 constants which may vary from line to line but do not depend on m or k. The case $m = 0$ has been established in Theorem 2. By (46), we have

$$\begin{aligned}\partial_t^m \partial_x^k \theta &= -\partial_t^{m-1}\partial_x^k(u\cdot\nabla\theta + \Lambda^\gamma\theta)\\ &= -\sum_{j=0}^{m-1}\sum_{l=0}^{k}\binom{m-1}{j}\binom{k}{l}\partial_t^j\partial_x^l u\cdot\partial_t^{m-1-j}\partial_x^{k-l+1}\theta - \partial_t^{m-1}\partial_x^k\Lambda^\gamma\theta.\end{aligned}$$

By (49), we have for some $C_1 > 0$ independent of (m,k),

$$\left\|\partial_t^{m-1}\partial_x^k\Lambda^\gamma\theta\right\|_{L_x^q} \leq C_1 \left\|\partial_t^{m-1}\partial_x^k\theta\right\|_{L_x^q}^{1-\frac{\gamma}{2}} \cdot \left\|\partial_t^{m-1}\partial_x^{k+2}\theta\right\|_{L_x^q}^{\frac{\gamma}{2}}.$$

By the Sobolev imbedding theorem, we have

$$\left\|\partial_t^{m-1-j}\partial_x^{l+1}\theta\right\|_{L_x^\infty} \leq \left\|\partial_t^{m-1-j}\partial_x^{l+1}\theta\right\|_{L_x^q} + \left\|\partial_t^{m-1-j}\partial_x^{l+2}\theta\right\|_{L_x^q}.$$

Now using the fact that $t \leq 1$, in terms of $C(m,k)$, we have

$$C(m,k) \leq C_1 \sum_{j=0}^{m-1} \sum_{l=0}^{k} \binom{m-1}{j} \binom{k}{l} C(j,l) \cdot (C(m-1-j, k-l+1) \\ + C(m-1-j, k-l+2)) + C_1 C(m-1,k)^{1-\frac{\gamma}{2}} C(m-1,k+2)^{\frac{\gamma}{2}}.$$

Now taking constants C_2, C_3, C_4 sufficiently large, it is easy to see that

$$C(m,k) \leq C_2^m C_3^{2m+k+1} F(2m+k),$$

where $F(0) \leq C_4$, and

$$F(n) = \sum_{0 \leqq n_1 \leqq n_2 \leqq n-1, n_1+n_2 \leq n} \frac{n!}{n_1! \cdot n_2!} F(n_1) F(n_2).$$

An easy application of Lemma 4 concludes the proof of the lemma. □

Finally the case $t > 1$ is not very different and we omit the details here.

9. Remark on the equation with periodic condition

We consider in this section the two-dimensional quasi-geostrophic equations on the torus:

$$\begin{cases} \theta_t + u\nabla\theta + (-\Delta)^{\gamma/2}\theta = 0, & \text{on } \mathbb{T}^2 \times (0,\infty), \\ \theta(0,x) = \theta_0(x), & x \in \mathbb{T}^2, \end{cases} \tag{51}$$

where $\mathbb{T}^2 = [0,1]^2$ and $\theta_0 \in L^{2/(\gamma-1)}(\mathbb{T}^2)$. As usual, the zero-average condition is assumed:

$$\int_{\mathbb{T}^2} \theta_0(x)\,\mathrm{d}x = 0.$$

For two periodic functions f, g defined on the torus $\mathbb{T}^2$ (that is, periodic in both variables with period 1), we define the convolution

$$f *_{\mathbb{T}} g(x) = \int_{\mathbb{T}^2} f(y) g(x-y)\,\mathrm{d}y.$$

Define the kernel on $\mathbb{T}^2$

$$\tilde{G}(t,x) = \sum_{m \in \mathbb{Z}^2} G(t, x+m) - 1.$$

Then it is easy to see that $\tilde{G}$ satisfies the zero-mean condition and

$$\tilde{G}(t+s,\cdot) = \tilde{G}(t,\cdot) *_{\mathbb{T}} \tilde{G}(s,\cdot). \tag{52}$$

Because θ_0 has zero mean and $\nabla \cdot u = 0$, (51) can be rewritten as

$$\theta(t) = \Theta(t) - B(\theta, \theta)(t), \tag{53}$$

where

$$\Theta(t) = \tilde{G} *_{\mathbb{T}} \theta_0, \quad B(v, w) := \int_0^t \nabla \tilde{G}(t-s, \cdot) *_{\mathbb{T}} (\mathcal{R}v \cdot w)(s, \cdot)\, \mathrm{d}s.$$

We claim that all the results in the previous sections still hold true. First we recall that we still have the L^p maximum principle for the quasi-geostrophic equations on the torus. To prove the corresponding results in Theorems 1, 2, and 3 and Corollary 1 it suffices to use the following lemma.

Lemma 10. *For any $\gamma \in [1, 2]$, $p \in [1, \infty]$, $k \geqq 0$, and $t \in (0, \infty)$, we have*

$$\|D_x^k \tilde{G}(t, \cdot)\|_{L_x^p(\mathbb{T}^2)} \leqq C^{k+1} k^{\frac{k}{\gamma}} t^{-\frac{k}{\gamma} - \frac{2}{\gamma}\left(1-\frac{1}{p}\right)} \mathrm{e}^{-t} \tag{54}$$

with a constant C independent of k.

Proof. Clearly, for $t \in (0, 1]$, it suffices to show

$$\|D_x^k \tilde{G}(t, \cdot)\|_{L_x^p(\mathbb{T}^2)} \leqq C^{k+1} k^{\frac{k}{\gamma}} t^{-\frac{k}{\gamma} - \frac{2}{\gamma}\left(1-\frac{1}{p}\right)}. \tag{55}$$

First we consider the case when $k = 0$ and $p < \infty$. Integration by parts gives

$$G(1, x) \leqq C(1 + |x|)^{-3}.$$

It is easy to see that

$$\begin{aligned}
(\tilde{G}(t, x))^p &= \left(\sum_{m \in \mathbb{Z}^2} G(t, x+m) - 1 \right)^p \\
&\leqq C \sum_{|m| \leqq 10} G(t, x+m)^p + C \left(\sum_{|m|>10} G(t, x+m) \right)^p + C.
\end{aligned}$$

Since the function $(1 + |x|)^{-3}$ is decreasing with respect to $|x|$, for $x \in [0, 1]^2$

$$\sum_{|m|>10} G(t, x+m) \leqq C \int_{|x|>2} t^{-2/\gamma} \left(1 + |x| t^{-1/\gamma}\right)^{-3} \mathrm{d}x \leqq C,$$

and we get

$$\|\tilde{G}(t, \cdot)\|^p_{L_x^p(\mathbb{T}^2)} \leqq C \|G(t, \cdot)\|^p_{L_x^p(\mathbb{R}^2)} + C \leqq C t^{-\frac{2}{\gamma}\left(1-\frac{1}{p}\right)}. \tag{56}$$

The cases $(k, p) = (0, \infty)$ and $k = 1$ are similar. For $k \geqq 2$, one can use Young's inequality and the identity

$$D_x^k \tilde{G}(t, \cdot) = D_x \tilde{G}\left(\frac{t}{2k}, \cdot\right) * \cdots * D_x \tilde{G}\left(\frac{t}{2k}, \cdot\right) * \tilde{G}\left(\frac{t}{2}, \cdot\right). \tag{57}$$

Next we consider the case when $t \geqq 1$. By Hölder's inequality, it suffices to prove (54) for $p = \infty$. Due to the definition of $\tilde{G}$,

$$\tilde{G}(t,x) = C \sum_{\xi \in 2\pi\mathbb{Z}^2 \setminus \{(0,0)\}} \mathrm{e}^{ix\xi} \mathrm{e}^{-|\xi|^\gamma t}.$$

Thus for $t \geqq 1/2$,

$$\|\tilde{G}(t,\cdot)\|_{L_x^\infty(\mathbb{T}^2)} \leqq C \sum_{\xi \in 2\pi\mathbb{Z}^2 \setminus \{(0,0)\}} \mathrm{e}^{-|\xi| t} \leqq C\mathrm{e}^{-\pi t}. \tag{58}$$

This yields (54) with $k = 0$. The case when $k = 1$ can be treated similarly, and we get

$$\|D_x \tilde{G}(t,\cdot)\|_{L_x^\infty(\mathbb{T}^2)} \leqq C\mathrm{e}^{-\pi t}. \tag{59}$$

For any integer $k \in [2, t)$, from (55), (58), Young's inequality, and the identity (57),

$$\|D_x^k \tilde{G}(t,\cdot)\|_{L_x^\infty(\mathbb{T}^2)} \leqq C^{k+1}(t/k)^{-\frac{k}{\gamma}} \mathrm{e}^{-\frac{\pi t}{2}} \leqq C^{k+1} k^{\frac{k}{\gamma}} t^{-\frac{k}{\gamma}-\frac{2}{\gamma}} \mathrm{e}^{-t}.$$

Similarly, for any integer $k \in [t, \infty)$, from (58), (59), Young's inequality and the identity (57),

$$\|D_x^k \tilde{G}(t,\cdot)\|_{L_x^\infty(\mathbb{T}^2)} \leqq \left(C\mathrm{e}^{-\frac{\pi t}{2k}}\right)^k \mathrm{e}^{-\frac{\pi t}{2}} \leqq C^{k+1} k^{\frac{k}{\gamma}} t^{-\frac{k}{\gamma}-\frac{2}{\gamma}} \mathrm{e}^{-t}.$$

The lemma is proved. □

We shall show that the solution along with all its derivatives decays *exponentially* as t goes to infinity. Recall the following decay estimate of θ (see, for example, [11]):

Lemma 11. *If $\theta_0 \in L^q$ for $q \in [1, \infty)$, we have exponential decay of $\|\theta\|_{L_x^q}$:*

$$\|\theta(t,\cdot)\|_{L_x^q} \leqq \|\theta_0\|_{L_x^q} \mathrm{e}^{-\lambda^\gamma t/q}, \tag{60}$$

where λ is a positive constant independent of γ and q.

For any $q \in (\frac{2}{\gamma-1}, \infty)$, denote $\varepsilon_1 = \min\{1, \lambda^\gamma t/q\}$.
Now let us state the main result of this section.

Theorem 6. *Let $\gamma \in (1, 2]$, $\theta_0 \in L^{\frac{2}{\gamma-1}}$, and $q \in (\frac{2}{\gamma-1}, \infty)$. Then the initial value problem for (51) has a unique mild solution $\theta(t,x)$ in $X_{q,\infty}^k$ for any integer $k \geqq 0$, and the following exponential decay in time estimate holds*

$$\max\left\{\|D_x^k\theta(t,\cdot)\|_{L_x^{2/(\gamma-1)}}, t^\alpha \|D_x^k \theta(t,\cdot)\|_{L_x^q}\right\} \leqq C^k \mathrm{e}^{-\varepsilon_1 \delta t} t^{-\frac{k}{\gamma}} k^k \tag{61}$$

with positive constants C and δ independent of k and t.

Proof. Thanks to (the corresponding version of) Theorem 2, we only have to prove (61) for $t \geqq 2$. By using Lemma 5, after a change of time variable $t \to t-1$, we may assume that $\theta_0 \in L^q \cap L^{2/(\gamma-1)}$. We now follow the proof of Theorem 2. Let $\check{R} = \|e^{\varepsilon_1 t}\Theta\|_{X^0_{q,\infty}}$, and $\check{\varepsilon} \in (0, \check{R})$ be a small positive number such that (26) holds with $\check{\varepsilon}$, $\check{R}$ in place of ε, R respectively. Choose $\delta \in (0, 1/3)$ sufficiently small, satisfying

$$(2\alpha\delta e)^{\alpha}(\varepsilon_1(1-3\delta))^{-\alpha}\|\theta_0\|_{L^q} \leqq \check{\varepsilon}. \tag{62}$$

For any fixed $t_1 > 0$, let $s = (1-\delta)t_1$. Let $\bar{\theta} \in X^0_{q,\infty}$ be the unique mild solution of (1) with initial data $\bar{\theta}_0 = \theta(s, \cdot)$. Let $\bar{\Theta}$ be the solution to the corresponding linear equation. Denote

$$\check{\theta} = e^{\varepsilon_1 t}\bar{\theta}, \quad \check{\Theta} = e^{\varepsilon_1 t}\bar{\Theta}, \quad \check{G} = e^{\varepsilon_1 t}\tilde{G}.$$

For any $k \geqq 0$, the L^p maximum principle and Lemma 10 imply

$$\|\check{\Theta}\|_{X^k_{q,\infty}} \leqq C^{k+1}k^k, \tag{63}$$

$$\|D^k_x \check{G}(t,\cdot)\|_{L^p_x(\mathbb{T}^2)} \leqq C^{k+1}k^{\frac{k}{\gamma}}t^{-\frac{k}{\gamma}-\frac{2}{\gamma}\left(1-\frac{1}{p}\right)} \tag{64}$$

with a constant C independent of k. From (60) and (62), we have

$$\begin{aligned}
\|t^{\alpha}\check{\Theta}\|_{L^\infty_t((0,2\delta t))L^q_x} &\leqq (2\delta t_1)^{\alpha}\|\check{\Theta}\|_{L^\infty_t((0,2\delta t_1))L^q_x} \\
&\leqq (2\delta t_1)^{\alpha}e^{2\varepsilon_1\delta t_1}\|\bar{\Theta}\|_{L^\infty_t((0,2\delta t_1))L^q_x} \\
&\leqq (2\delta t_1)^{\alpha}e^{2\varepsilon_1\delta t_1}\|\theta(s)\|_{L^q_x} \\
&\leqq (2\delta t_1)^{\alpha}e^{2\varepsilon_1\delta t_1-\varepsilon_1(1-\delta)t_1}\|\theta_0\|_{L^q_x} \\
&\leqq (2\alpha\delta e)^{\alpha}(\varepsilon_1(1-3\delta))^{-\alpha}\|\theta_0\|_{L^q_x} \leqq \check{\varepsilon}.
\end{aligned} \tag{65}$$

Due to (53), it holds that

$$\check{\theta}(t) = \check{\Theta}(t) - \check{B}(\check{\theta}, \check{\theta})(t), \tag{66}$$

where

$$\check{B}(v, w) := \int_0^t e^{-\varepsilon_1 s}\nabla\check{G}(t-s,\cdot) *_{\mathbb{T}} (\mathcal{R}v \cdot w)(s,\cdot)\,ds.$$

Notice that $\check{\Theta}$ and $\check{G}$ verify the same estimates (63) and (64) as Θ and G in the proof of Proposition 1. Also the extra factor $e^{-\varepsilon_1 s}$ is less than 1. These facts, together with (65) and (66) allow us to reproduce the contraction argument in the proof of Proposition 1 and the estimate of the analyticity rate in the proof of Proposition 2, and we get

$$\|\check{\theta}\|_{X^k_{q,2\delta t_1}} \leqq C^{k+1}k^k. \tag{67}$$

The estimate above can easily conclude (61), and the theorem is proved. □

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Division of Applied Mathematics, Brown University,
182 George Street, Box F, Providence, RI 02912, USA.
e-mail: hdong@brown.edu

and

School of Mathematics, Institute for Advanced Study,
Einstein Drive, Princeton, NJ 08540, USA.
e-mail: dongli@math.ias.edu

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